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STABLY ERGODIC DIFFEOMORPHISMS WHICH ARE NOT PARTIALLY HYPERBOLIC

BY

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ABSTRACT

We show stable ergodicity of a class of conservative diffeomorphisms of $\mathbb{T}^n$ which do not have any hyperbolic invariant subbundle. Moreover, the uniqueness of SRB (Sinai–Ruelle–Bowen) measure for non-conservative C^1 perturbations of such diffeomorphisms is verified. This class strictly contains non-partially hyperbolic robustly transitive diffeomorphisms constructed by Bonatti–Viana [4] and so we answer the question posed there on the stable ergodicity of such systems.

1. Introduction

A main objective of Dynamical Systems is to answer the following questions:

(1) What main topological or metric properties are satisfied by the majority of dynamical systems?
(2) Under which conditions do such properties persist after small perturbation of the system?

Ergodicity is a basic feature for conservative dynamical systems that yields the description of the average time spent by typical orbits in different regions of the phase space. For non-conservative systems, existence and uniqueness (or just finiteness) of SRB (Sinai–Ruelle–Bowen) measures plays a similar role.

A few years ago Palis [11] conjectured: Every system can be C^r approximated, any $r \geq 1$, by one having finitely many SRB measures with their basins covering a full Lebesgue measure of the phase space.

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In other words, one expects that for a "majority" of diffeomorphisms, the average time spent by typical orbits in different regions of the phase space is described by at most a finite number of measures.

In this direction, in [1] the authors show the existence of finitely many SRB measures with basins covering a full Lebesgue measure of the ambient manifold, for a large class of partially hyperbolic systems and more generally for systems displaying a dominated splitting.

Definition 1.1: Let M be a compact manifold and $f\colon M \to M$ a C^1 diffeomorphism. We say that the splitting $TM = E^{cs} \oplus E^{cu}$ is dominated if it is Df invariant and there exist $C > 0$ and $\lambda < 1$ such that

$$\|Df|E^{cs}_x\|.\|Df^{-1}|E^{cu}_{f(x)}\| \leq \lambda \quad \text{for all } x \in M.$$

A diffeomorphism is called **partially hyperbolic** if the tangent bundle admits a dominated splitting and at least one of the sub-bundles E^{cu} or E^{cs} is uniformly hyperbolic.

In this work we take over from where [1] and [4] left off, to provide sufficient conditions for stable ergodicity (conservative case) and uniqueness of SRB measures (general dissipative case). The main novelty of our results is that we prove that very weak hyperbolicity (dominated splitting) may suffice for stable ergodicity.

Definition 1.2: A C^2 conservative diffeomorphism f is stably ergodic if any C^2 conservative diffeomorphism g nearby to f in C^1 topology is also ergodic.

Let us recall that Anosov [2] proved that every C^2 globally uniformly hyperbolic volume preserving diffeomorphism is ergodic. In the corresponding dissipative setting, Sinai [16] proved existence and uniqueness of the SRB measures. More recently, Pugh, Shub and other collaborators obtained stable ergodicity for a large class of volume-preserving diffeomorphisms, assuming a dominated splitting $E^s \oplus E^c \oplus E^u$ exists, where E^u is uniformly expanding and E^s is uniformly contracting. (See [5] and references therein.)

Here we drop any assumption about existence of uniformly hyperbolic subbundles. Before we give the precise statement of our results, let us comment on some of the main new difficulties in our context.

A classical strategy for proving ergodicity, going back to Hopf, is by propagating statistical behaviors of orbits along invariant (stable and unstable) foliations. In the context treated in [5] and related publications there are integral foliations tangent to E^s and E^u. One says that the map is accessible (resp. essentially accessible) if any (resp. almost all) two points may be joined by a path consisting of consecutive segments, which are part of stable or unstable foliations. Accessibility or at least essential accessibility is a key ingredient for proving ergodicity.

Such a strategy does not make sense in our case, since systems with a dominated splitting need not have invariant foliations. As we shall explain in more detail later, we handle this by first proving non-uniform hyperbolicity: all Lyapunov exponents non-zero almost everywhere. This places us in the setting of [1] and [4] and we prove uniqueness of the SRB measure constructed there. Restricted to conservative diffeomorphisms this gives stable ergodicity.

1.2. Statement of Results. Let M be a compact manifold endowed with a volume form ω. Let $f\colon M \to M$ be a diffeomorphism. Given an f-invariant Borel probability measure μ, we call basin of μ the set $B(\mu)$ of $x \in M$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^i(x)) = \int \phi d\mu \quad \text{for every } \phi \in C^0(M)$$

and say that μ is a physical or SRB (Sinai–Ruelle–Bowen) measure for f if $B(\mu)$ has positive Lebesgue measure.

Let us introduce the class of diffeomorphisms for which our results apply. The class $\mathcal{V} \subset \mathrm{Diff}^1(\mathbb{T}^n)$ under consideration consists of diffeomorphisms which are deformations of an Anosov diffeomorphism. To define $\mathcal{V}$, let f_0 be a linear Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$ (in fact, we need f_0 only to be an Anosov diffeomorphism on $M = \mathbb{T}^n$ whose foliations lifted to $\mathbb{R}^n$ are global graphs of C^1 functions over the corresponding invariant subbundles). Denote by $TM = E_0^s \oplus E_0^u$ the hyperbolic splitting for f_0 with dim $(E_0^s) = s$, dim $(E_0^u) = u$ and let $V = \bigcup V_i$ be a finite union of small pairwise disjoint balls in $\mathbb{T}^n$. We suppose that f_0 has at least one fixed point outside V. By definition $f \in \mathcal{V}$ if it satisfies the following C^1 open conditions:

(1) TM admits a dominated decomposition and there exist small continuous cone fields C^{cu} and C^{cs} invariant for Df and Df^{-1} containing respectively E_0^u and E_0^s.

(2) f is C^1 close to f_0 in the complement of V, so that for $x \notin V$ there is $\sigma < 1$:

$$\|(Df|T_xD^{cu})^{-1}\| < \sigma \quad \text{and} \quad \|Df|T_xD^{cs}\| < \sigma.$$

(3) There exists some small $\delta_0 > 0$ such that for $x \in V$

$$\|(Df|T_xD^{cu})^{-1}\| < 1 + \delta_0 \quad \text{and} \quad \|(Df|T_xD^{cs}\| < 1 + \delta_0$$

where D^{cu} and D^{cs} are disks tangent to C^{cu} and C^{cs}.

THEOREM 1: *Every $f \in \mathcal{V} \cap \mathrm{Diff}^2_\omega(\mathbb{T}^n)$ is stably ergodic.*

For non-conservative diffeomorphisms in $\mathcal{V}$ we prove uniqueness of SRB measure. In fact we assume the diffeomorphisms are **volume hyperbolic** (in the conservative case this is automatic from the domination property):

Definition 1.3: Let $f\colon M \to M$ be a C^1 diffeomorphism and $TM = E^1 \oplus E^2$; we say that this decomposition is volume hyperbolic if for some $C > 0$ and $\lambda < 1$

$$|\det(Df^n(x)|E^1)| \leq C\lambda^n \quad \text{and} \quad |\det(Df^{-n}(x)|E^2)| \leq C\lambda^n.$$

THEOREM 2: *Any $f \in \mathcal{V} \cap \mathrm{Diff}^2(\mathbb{T}^n)$ for which the dominated decomposition $TM = E^{cs} \oplus E^{cu}$ is volume hyperbolic has a unique SRB measure with a full Lebesgue measure basin.*

In Theorem 2, volume hyperbolicity is crucial for proving non-uniform hyperbolicity. Roughly speaking, by means of this property and a good control of the invariant leaves of f_0, typical orbits do not stay a long time in V and so their asymptotic behavior is mostly influenced by the hyperbolicity condition 2.

1.2. OUTLINE OF THE PAPER. The paper is organized as follows. In Section 2, we give some preliminary definitions which will be used in the rest of the paper. In Section 3, we exhibit an explicit open set of diffeomorphisms which satisfy the hypothesis of our results and yet have no uniformly hyperbolic (expanding or contracting) invariant subbundle. The complete proof of Theorems 1 and 2 occupies Sections 4–7.

In Section 4 we analyze the geometry of the basins of the SRB measures constructed in [1] for systems with dominated splitting. In Section 5 we deduce our main results from certain facts that are proved subsequently. Indeed in Section 6 we prove non-uniform hyperbolicity for almost all points of any submanifold with *good geometry*. In particular, Lebesgue almost all points of the ambient manifold satisfy the non-uniform hyperbolicity conditions. The other important ingredient, proved in Section 7, is absolute continuity of local stable/unstable lamination obtained from Section 6. It is worthwhile pointing out that, since we have to deal with possibly non-regular points, the conclusions of these two last sections cannot be deduced from general arguments in Pesin theory. In fact, good control of the angle, given by the domination condition, is crucial to our approach.

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2. Preliminaries

In this section we consider some ways of relaxing uniform hyperbolicity.

2.1. NON-UNIFORM HYPERBOLICITY. This approach is due to Pesin [13] and it refers to diffeomorphisms with non-zero Lyapunov exponents in a full measure subset of phase space. Recall that λ is a Lyapunov exponent at x if

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda$$

for some vector $v \in T_x M$. By Oseledets' theorem Lyapunov exponents exist for a total probability subset of M.

To construct SRB measures for systems with a dominated splitting, by the methods in [1] we need to verify "non-uniform hyperbolicity" in a total Lebesgue measure set in the following sense. There exist a positive Lebesgue measure set H and $c_0 > 0$ such for $x \in H$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df|E^{cu}_{f^j(x)})^{-1}\| \leq -c_0 \tag{1}$$

and also

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df|E^{cs}_{f^j(x)}\| \leq -c_0. \tag{2}$$

In our setting we prove that H has full Lebesgue measure. This is a crucial step for the proof of Theorem 1. We mention that the above conditions imply non-zero Lyapunov exponents. Let us just mention that in the Pesin theory, some invariant measure is fixed and non-uniformly hyperbolic systems refer to ones without zero Lyapunov exponent in a total measure set. But, we are working with the Lebesgue measure which is not invariant for non-conservative diffeomorphisms of $\mathcal{V}$. In this paper, by non-uniform hyperbolicity we refer to the above conditions.

2.2. Dominated Splitting. This approach was much used by Mañé in his proof of the stability conjecture [9]. Suppose that $TM = E^{cs} \prec E^{cu}$ is a dominated splitting of TM. By this notation we emphasize that E^{cu} is dominating E^{cs}. Whenever we have a dominated splitting on TM as defined in 1.1 there are two cone fields C^{cu} (**center unstable**) and C^{cs} (**center stable**) with the following properties:

$$C^{cu}_a(x) = \{v_1 + v_2 \in E^{cs} \oplus E^{cu}; \|v_1\| \leq a\|v_2\|\}, \quad Df(C^{cu}_a(x)) \subset C^{cu}_{\lambda a}(f(x)),$$
$$C^{cs}_a(x) = \{v_1 + v_2 \in E^{cs} \oplus E^{cu}; \|v_2\| \leq a\|v_1\|\}, \quad Df^{-1}(C^{cs}_a(x)) \subset C^{cs}_{\lambda a}(f^{-1}(x)).$$

To verify non-uniform hyperbolicity for the diffeomorphisms in Theorems 1 and 2, we use the volume hyperbolicity property defined in Definition 1.3.

For non-trivial examples of diffeomorphisms with volume hyperbolicity property we point out that (see [3]) a dominated splitting $TM = E^1 \prec E^2$ for any C^1 conservative diffeomorphism is volume hyperbolic. From this and the continuity of $\det Df$, we conclude the following:

Corollary 2.1: *For any $f \in \mathcal{V} \cap \mathrm{Diff}^2_\omega(\mathbb{T}^n)$, there exist $\sigma_1 > 1$ and $C > 0$ such that*

$$|\det(Df^n(x)|T_x(D^{cs}))| \leq C\sigma_1^{-n} \quad \text{and} \quad |\det(Df^{-n}(x)|T_x(D^{cu}))| \leq C\sigma_1^{-n}$$

where D^{cs} and D^{cu} are disks tangent to C^{cs} and C^{cu}.

3. Robustly transitive diffeomorphisms of $\mathbb{T}^n$

Here we give a C^1 open set of diffeomorphisms that satisfy the hypothesis of Theorems 1 and 2. A diffeomorphism f is called **robustly transitive** if any C^1 nearby diffeomorphism to f is also transitive. The first non-partially hyperbolic and robustly transitive example is constructed in [4] on $\mathbb{T}^4$. We will construct the example of robustly transitive diffeomorphisms without hyperbolic sub-bundles in higher than 4 dimensions. Let f_0 be a volume preserving linear Anosov diffeomorphism on the $\mathbb{T}^n$ for which

$$T_x(\mathbb{T}^n) = \mathbb{R}^n = E^s_1 \prec E^s_2 \prec \cdots \prec E^s_{n-2} \prec E^u$$

where $\dim(E^u) = 2$ and $\dim(E^s_i) = 1$ and E^u is uniformly expanding and all E^s_i are uniformly contracting.

We may suppose that f_0 has fixed points $p_1, p_2, \ldots, p_{n-2}$. Let $V = \bigcup B(p_i, \delta)$ be a union of balls centered at p_i and radius sufficiently small $\delta > 0$. By iteration, we may also suppose that f_0 has a fixed point out of V. The idea is to deform the Anosov diffeomorphism inside V, passing first through a pitchfork bifurcation along $(E_i^s \oplus E_{i+1}^s)(p_i)$ inside $B_i = B(p_i, \delta)$ and then another deformation (see Figure 1) to get a complex eigenvalue for a fixed point near to p_i.

More precisely, first we modify along $E_i^s(p_i) \oplus E_{i+1}^s(p_i)$ for $1 \leq i \leq n-3$ until the stable index of p_i drops one and two fixed points q_i, r_i are created. These new fixed points have stable index equal to $n-2$. In the next step we mix the two contracting sub-bundles of $T_{q_i}M$ corresponding to $E_i^s(q_i)$ and $E_{i+1}^s(q_i)$ and get a complex eigenvalue. These modifications can be done by an isotopy and in a way to obtain volume preserving diffeomorphisms (see [4]).

After these deformations we get a new diffeomorphism which we also call f and we have the following Df invariant decomposition for the tangent bundle of q_i:

$$T_{q_i}M = E_1 \prec \cdots \prec E_{i-1} \prec F_i \prec \cdots \prec E^u$$

where E_i is one-dimensional, E^u is two-dimensional and uniformly expanding and F_i is the two-dimensional sub-bundle corresponding to the complex eigenvalue. Finally, we do the same for p_{n-2}, but in its unstable direction. That is, after the modifications along the unstable sub-bundle of p_{n-2} we get a new fixed point q_{n-2} such that

$$T_{q_{n-2}}M = E_1 \prec \cdots \prec E_{n-2}^s \prec F_{n-2}$$

and F_{n-2} is the sub-bundle corresponding to the complex expanding eigenvalue of q_{n-2}.

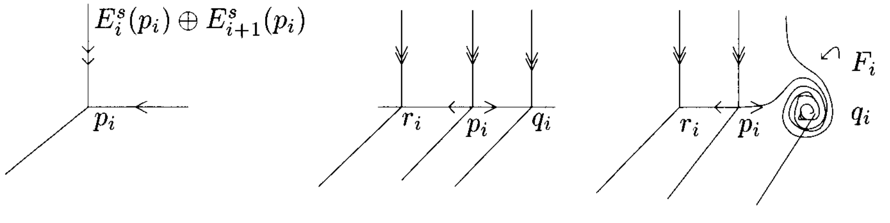


Figure 1. Deformation of Anosov.

In this way we get a C^1 open set $\tilde{\mathcal{V}}$ of diffeomorphisms satisfying the conditions (1)–(3) mentioned in the Introduction. Another important fact is about the mentioned hyperbolic fixed point outside V.

We supposed that there exists a hyperbolic fixed point q of f_0 outside V with stable index $s = n - 2$ in our example. For any $f \in \tilde{\mathcal{V}}$, as f is C^1 near to Anosov diffeomorphism outside V, it has a fixed point outside V which is the continuation of q, which we will also call q.

It is easy to see that the stable manifold of the continuation of q intersects any disk tangent to C^{cu} with radius more than ϵ_0, for some small $\epsilon_0 > 0$. The similar thing for the unstable manifold and disks tangent to C^{cs} happens. This is just because of the denseness of stable and unstable leaves of f_0. Indeed, take a compact part of $W^s(q, f_0)$ to be ϵ_0-dense. Now, if V is small enough this compact part of the stable manifold continues to be a part of the stable manifold for the continuation of q.

Remark 3.1: Clearly, the last item above is satisfied for $f \in \mathcal{V}$ of Theorems 1 and 2, as V is small enough. Namely, any $f \in \mathcal{V}$ has a hyperbolic fixed point $q \notin V$ such that its stable and unstable manifolds are ϵ_0-dense. In fact, later we prove the density of these invariant manifolds (see Proposition 5.1).

LEMMA 3.2: *$f \in \tilde{\mathcal{V}}$ is robustly transitive.*

Proof: The proof goes as in the $\mathbb{T}^4$ case in [4, Lemma 6.8] and we just remember the steps. The main idea to prove robust transitivity is to show the robust density of the stable and unstable manifold of an hyperbolic fixed point. We show the density of invariant manifolds of q defined in Remark 3.1 (see Proposition 5.1).

Let U and V be two open subsets. Using the λ-lemma and the density of invariant manifolds of q we intersect some iterate of U with V and get transitivity of f. ∎

LEMMA 3.3: *$f \in \tilde{\mathcal{V}}$ is not partially hyperbolic.*

Proof: This is just because of the definition of partially hyperbolic systems. f is partially hyperbolic if $TM = E^s \oplus E^c \oplus E^u$ is a decomposition into continuous sub-bundles where at least two of them are non-zero where E^s and E^u are respectively uniformly contracting and expanding. Suppose that f is partially hyperbolic. By the continuity of the decomposition of TM and existence of a dense orbit by Lemma 3.2, the dimension of E^s and E^u is constant.

We claim that $\dim(E^s) = n - 2$ and this results in a contradiction, because in $T_{p_i}M$ there do not exist $n - 2$ contracting invariant directions. To prove the claim observe that if we suppose that $\dim(E^s) = j < n-2$, then by the decomposition of $T_{q_j}M$

$$T_{q_j}M = E_1 \prec \cdots \prec E_{j-1} \prec F_j \prec \cdots \prec E^u.$$

By definition, $E^s(q_j)$ must contain $E_1 \oplus \cdots \oplus E_{j-1}$ and then, as F_j does not have any invariant sub-bundle, we conclude that $\dim(E^s(q_j)) \geq j+1$ and this is a contradiction, because $\dim(E^s) = j$. This shows that $\dim(E^s) = 0$. By investigating $T_{p_{n-2}}M$, it is obvious that TM cannot have invariant unstable sub-bundles, too. ∎

4. *cu*-Gibbs measures

Pesin and Sinai defined and constructed Gibbs measures for partially hyperbolic dynamical systems. They defined Gibbs measure as measures which are absolutely continuous with respect to Lebesgue measure along the unstable foliation of a partially hyperbolic diffeomorphism ([12]).

For systems with a dominated splitting $TM = E^{cs} \prec E^{cu}$, one defines *cu*-Gibbs measure as an invariant measure with positive Lyapunov exponents along E^{cu} and absolutely continuous with respect to the Lebesgue measure along the Pesin local unstable manifolds tangent to E^{cu}. In [1], *cu*-Gibbs measure for the systems with dominated splitting and having non-uniform hyperbolicity property is constructed. However, they show that under a technical condition called "simultaneous hyperbolic times" the *cu*-Gibbs measures are in fact SRB measures.

In this paper we prove that any $f \in \mathcal{V}$ is non-uniformly hyperbolic as required in [1] and, without verifying the simultaneous hyperbolic times condition, in Appendix B prove that the constructed *cu*-Gibbs measures are SRB. In fact we prove the following theorem.

For a submanifold D of M, by Leb_D we mean the Lebesgue measure of D and by $A \subset B$ (μ mod-0) we mean $\mu(A \setminus B) = 0$.

THEOREM 3: *Let f be as in Theorem 2. Then $M = \bigcup B(\mu_i)$ (mod-0) where μ_i are ergodic SRB measures and for each μ_i there exists a disk D_i^∞ tangent to a center-unstable cone field such that $D_i^\infty \subset B(\mu_i)$ ($Leb_{(D_i^\infty)}$ mod-0).*

Let us recall briefly the construction of *cu*-Gibbs measures. Fix a C^2 disk tangent to C^{cu} at every point of it and intersecting H (the set of points having non-uniformly hyperbolic behavior) in a positive Lebesgue measure, where by measure we mean the Lebesgue measure of the disk. Now consider the sequence

μ_n of averages of forward iterates of Lebesgue measure restricted to such a disk. A main idea is to decompose μ_n as a sum of two measures, denoted by ν_n and η_n, such that ν_n is uniformly absolutely continuous on iterates of the disk and has total mass uniformly bounded away from zero for all large n. Taking a sub-sequence if necessary we may suppose that μ_n converges to μ and ν_n to ν. Finally, we show that absolute continuity passes to ν, the limit of ν_n. More precisely:

PROPOSITION 4.1 ([1]): *There exists a cylinder $\mathcal{C}$ (a diffeomorphic image of product of two balls B^u and B^s of dimensions $\dim(E^{cu})$ and $\dim(E^{cs})$ in M) and a family $\mathcal{K}_\infty$ of disjoint disks contained in $\mathcal{C}$ which are graphs over B^u such that:*

1. *The union of all the disks in $\mathcal{K}_\infty$ has positive ν measure.*
2. *The restriction of ν to that union has absolutely continuous conditional measure along the disks in $\mathcal{K}_\infty$.*

So we have $\mu = \nu + \eta$, where ν is absolutely continuous with a bounded away from zero Radon–Nikodym derivative along a family of cu-disks. In this way we conclude that there exist disks γ where Leb_γ-almost every point in γ is regular and, by absolute continuity of the stable manifolds "for regular points", one gets a μ positive measure set in the same ergodic component of μ. Normalizing the restriction of μ to the ergodic component above, we get an ergodic invariant probability measure μ^*.

As the conditional measure of μ with respect to $\mathcal{K}_\infty$ is the sum of the conditional measures of ν and η, we conclude the following:

LEMMA 4.2: *There exists a disk D^∞ in $\mathcal{K}_\infty$ such that Leb_{D^∞}-almost every point of D^∞ belongs to the basin of μ^*.*

By Proposition 6.4 in [1], $M = \bigcup B(\mu_i)$ (mod -0), where μ_i's are cu-Gibbs ergodic measures. In fact these measures are also SRB measure, as we show in Appendix B. By the above lemma the proof of Theorem 3 is complete.

5. Uniqueness of SRB measures

In this section we prove Theorem 1 and Theorem 2 using some facts which we prove in the next sections. Let μ_i be as in Theorem 3. We prove that for $f \in \mathcal{V}$, $B(\mu_i) \cap B(\mu_j) \neq \emptyset$ for all $i \neq j$. But as μ_i's are ergodic so they are the same one. Let q be the fixed hyperbolic point of f mentioned in Remark 3.1.

PROPOSITION 5.1: *The global stable manifold of q is dense and intersects transversally each D_i^∞.*

Proof: To prove Proposition 5.1 we claim that some iterate of D_i^∞ contains a disk tangent to a center-unstable cone field with radius more than ϵ_0 of which also almost every point belongs to $B(\mu_i)$. This proves the Proposition, because of the ϵ_0-density of the $W^s(q)$. (See Remark 3.1.)

To prove the above claim consider a lift $\tilde{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ of f and let π_u be the projection along the stable foliation of f_0 (the Anosov one) from $\mathbb{R}^n$ to $\mathbb{R}^u$. As D_i^∞ is tangent to cone field C^{cu} we may consider a global graph Γ of a C^1 function γ: $\mathbb{R}^u \to \mathbb{R}^s, ||D\gamma|| \leq \epsilon$ (ϵ is the angle of the cone field) which contains D_i^∞. Let $\Gamma_n := \tilde{f}^n(\Gamma)$. Each Γ_n is the graph of a C^1 function with small norm of derivative. This is because $f^n(\Gamma)$ is a proper embedding of $\mathbb{R}^u$ in $\mathbb{R}^n$ whose tangent space at every point is in C^{cu} and C^{cu} is forward invariant.

Now as Df expands the area of disks in the center unstable direction, by arguments of [4, Lemma 6.8] there exists some point x_0 in $f^{n_0}(D_i^\infty)$ such that its positive orbit never intersects V, so any small disk in Γ_{n_0} around x_0 will have some iterate containing a disk with radius at least ϵ_0. (See Remark 3.1 for ϵ_0.) In this way we prove the density of $W^s(q)$. If U is any open set, just consider a center-unstable disk D, in the intersection of U and an unstable leaf of f_0 and argue as above, substituting D_i^∞ by D. The density of $W^u(q)$ comes out by the similar method.

Now observe that because of the invariance of continuous cone field C^{cs}, the global stable manifold of q is tangent to C^{cs} at any point and consequently the intersection of $W^s(q)$ and D_i^∞ is transversal. ∎

Using the λ-lemma, for n large enough $f^n(D_i^\infty)$ and $W^u(q)$ are C^1 near enough. On the other hand, in Section 6 (Corollary 6.8) we prove that almost every point of $W^u(q)$ has a local stable manifold. This implies that there exists $S \subset W^u(q)$ with $Leb(S) > 0$ such that for all $x \in S$ the size of $W^s_{loc}(x)$ is uniformly bounded away from zero and $W^s_{loc}(x)$ intersects $f^n(\mathcal{D}_i^\infty)$ for n large enough. We need an absolute continuity property proved in Section 7 to conclude the following:

$$Leb_{f^n(\mathcal{D}_i^\infty)}\left(\bigcup_{x\in S} W^s_{loc}(x) \cap B(\mu_i) \cap f^n(\mathcal{D}_i^\infty)\right) > 0.$$

We would get the same thing for μ_j and this enables us to find at least two points x, y respectively in $B(\mu_i)$ and $B(\mu_j)$ such that they are in the local stable manifold of the same point in S (see Figure 2). This means

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\phi(f^i(x))=\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\phi(f^i(y))\quad\text{for every }\phi\in C^0(M),$$

and consequently $B(\mu_i)\cap B(\mu_j)\neq\emptyset$, which implies $\mu_i=\mu_j$. We have proved that the decomposition of $\mathbb{T}^n$ (mod 0) by the basin of SRB measures contains a unique element (mod 0) or there exists just one SRB measure whose basin has full Lebesgue measure.

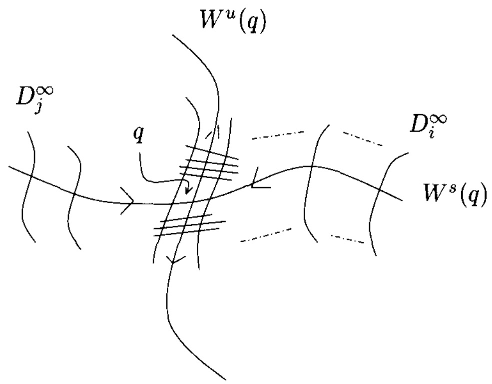


Figure 2. Intersecting basins via local stable manifolds.

Now let us explain how to conclude Theorem 1 from Theorem 2. If f preserves the Lebesgue measure, the dominated splitting of the tangent bundle is volume hyperbolic (see Preliminary). So by Theorem 2, for almost all points

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\phi(f^i(x))=\int\phi d\mu\quad\text{for every }\phi\in C^0(M)$$

and immediately we have ergodicity of Lebesgue measure, completing the proof of Theorem 1.

Remark: In Theorem 2 we prove the uniqueness of the SRB measures. The unique SRB measure is absolutely continuous along disks which are unstable manifolds corresponding to positive Lyapunov exponents. By [8] one has the following:

$$h_\mu(f)=\sum\lambda_i^+\quad\text{where }\lambda_i^+=\max\{0,\lambda_i\},$$

where λ_i are the Lyapunov exponents of the ergodic measure μ. In fact, as the basin of the physical measure constructed in Theorem 2 occupies a total

Lebesgue measure set of manifold, it will be the unique measure among the ergodic measures with non-zero Lyapunov exponents which satisfy Pesin's formula.

We observe that with the same method with which we have proved the uniqueness of SRB measures, one also can show that μ is the unique ergodic measure satisfying Pesin's equality and having u ($= \dim E^u$) positive Lyapunov exponents. Then by the ergodic decomposition theorem it is the unique invariant probability with the mentioned properties. So the following question is interesting:

QUESTION 1: *Does any f as in Theorem 2 have only one measure satisfying Pesin's equality?*

6. Non-uniform hyperbolicity

In this section we prove that $f \in \mathcal{V}$ is non-uniformly hyperbolic. In fact, we use the mechanism to prove the non-uniform hyperbolicity presented in [1] and prove the non-uniform hyperbolicity property for almost all points of many submanifolds. In particular, we prove the non-uniform hyperbolicity for almost all points of the unstable manifold of q, the fixed point of Remark 3.1.

Let W be a u-dimensional submanifold of $\mathbb{T}^n$ and π the natural projection from $\mathbb{R}^n$ to $\mathbb{T}^n$. We call W dynamically flat according to the following definition.

Definition 6.1: W is dynamically flat if for $\widetilde{W_n}$, any lift of $f^n(W)$ to $\mathbb{R}^n$, $Leb(\widetilde{W_n} \cap K) \leq C$ where K is any unit cube in $\mathbb{R}^n$ and C is a constant depending only on f.

LEMMA 6.2: *$W^u(q)$ is dynamically flat.*

Proof: Consider $\mathcal{F}_0(q)$, the leaf of unstable foliation of f_0 which passes through q, and let $\mathcal{F}_n = f^n(\mathcal{F}_0(q))$. As $\mathcal{F}_0$ is a leaf of a linear Anosov diffeomorphism, any lift of it to $\mathbb{R}^n$ will be a u-affine subspace and is a proper image of $\mathbb{R}^u$ to $\mathbb{R}^n$. By invariance of the thin cone field C^{cu} we conclude that the tangent space of any lift of $\mathcal{F}_n$, which we also call $\mathcal{F}_n$, at every point is in C^{cu} and it is also the proper image of $\mathbb{R}^u$. In this way, for any unit cube K, $\mathcal{F}_n \cap K$ can be seen as the graph of a C^1 function with u-dimensional base of the cube as its domain. This function has a small norm of derivative which is independent of cube K and n; this is because its graph is tangent to C^{cu}. So $\mathcal{F}_n \cap K$ has a uniformly bounded area (with respect to Lebesgue measure of $\mathcal{F}_n$) and this is what we want, because the intersection of the unstable manifold of q with K is contained in the limit of $\mathcal{F}_n \cap K$. ∎

PROPOSITION 6.3: *Let W be a dynamically flat submanifold and f satisfying the hypothesis of Theorem 2. Then every small disk in W contains a Lebesgue total measure (Lebesgue measure of W) subset for which*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|(Df|E^{cs}_{f^j(x)})\| \leq -c_0,$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|(Df|E^{cu}_{f^j(x)})^{-1}\| \leq -c_0,$$

where $c_0 > 0$.

Proof: Here we use the same arguments of [1] and prove that:

LEMMA 6.4: *There exists $\epsilon > 0$ and a total Lebesgue measure subset of any small disk D in W such that $\#\{0 \leq j < n : f^j(x) \notin V\} \geq \epsilon n$ for every large n.*

Proof: We choose a partition in domains $B_1, B_2, \dots, B_{p+1} = V$ of $\mathbb{T}^n$ such that there exist K_i, L_i with $B_i \in \pi(K_i)$ and $f(B_i) \in \pi(L_i)$, where K_i, L_i are finite open cubes in $\mathbb{R}^n$, and estimate the Lebesgue measure of the sets $[\underline{i}]$ where $\underline{i}$ is an array $(i_0, i_1, \dots, i_{n-1})$ and $[\underline{i}]$ is defined as the set of points in D such that $f^j(x) \in B_{i_j}$ for $0 \leq j < n$. In fact, we prove the following lemma. Let σ_1 be as in Corollary 2.1; then:

LEMMA 6.6: *$Leb([\underline{i}]) \leq C\sigma_1^{-n}$ (where C is a constant depending only on f).*

Proof: By the choice of B_i and induction we have that $f^j([\underline{i}]) \in \pi(\widetilde{W}_n \cap L_{i_{j-1}})$, where $\widetilde{W_n}$ is a lift of $f^n(W)$ to $\mathbb{R}^n$.

To conclude the lemma we use the area expanding (Corollary 2.1) property along disks tangent to the center-unstable cone field and the fact that the intersection of $\widetilde{W_n}$ with a unit cube has a uniformly bounded volume. By induction

$$Leb([\underline{i}]) \leq \sigma_1^{-n} Leb(f^n([\underline{i}]) \leq \sigma_1^{-n} Leb(\widetilde{W} \cap L_{i_{n-1}}) \leq C\sigma_1^{-n}. \quad ∎$$

Now we show how to prove Lemma 6.4. Let $g(\underline{i})$ be the number of values $0 \leq j \leq n-1$ for which $i_j \leq p$. We note that the total number of arrays with $g(\underline{i}) \leq \epsilon n$ is bounded by

$$\sum_{k \leq \epsilon n} \binom{n}{k} p^k \leq \sum_{k \leq \epsilon n} \binom{n}{k} p^{\epsilon n}$$

and applying Stirling's formula yields that it is bounded by $e^{\beta_0 n}p^{\epsilon n}$ (β_0 goes to zero as ϵ goes to zero). So the union of the sets $[\underline{i}]$ for which $g(\underline{i}) \leq \epsilon n$ has Lebesgue measure less than $Ce^{\beta_0 n}p^{\epsilon n}\sigma_1^{-n}$. Choosing ϵ small enough such that $e^{\beta_0}p^{\epsilon} < \sigma_1$, we are in the setting of the Borel–Cantelli lemma and Lemma 6.4 is proved. ∎

By this lemma, we conclude that almost all points in a dynamically flat disk spend a definite fraction of their orbit outside V, which is a **bad region** for uniform hyperbolicity.

To prove Proposition 6.3 it is enough to take $c_0 = -\log(\sigma^{\epsilon}(1+\delta_0)^{1-\epsilon})$, and δ_0 small enough guarantees that $c_0 > 0$. ∎

COROLLARY 6.6: *Any $f \in \mathcal{V}$ satisfies the non-uniform hyperbolicity condition as defined in Section 2.*

Proof: Consider a foliation by the unstable leaves of f_0 (of course, it is not invariant for f). Any leaf of this foliation lifted to $\mathbb{R}^n$, by our hypothesis, is the graph of a C^1 function defined on the corresponding sub-bundle. By the invariance of cone fields we can easily prove that these leaves are dynamically flat and then use Proposition 6.3 to prove the corollary. ∎

By Lemma 6.2 we conclude that:

COROLLARY 6.7: *Almost all points of the local unstable manifold of q satisfy the non-uniform hyperbolicity property.*

For any x satisfying the conclusion of Proposition 6.3, there exists $N(x)$ such that for $n \geq N$

$$\prod_{i=0}^{n-1} \|Df_{|E^{cs}(f^i(x))}\| \leq \lambda^n,$$

where λ is slightly larger than $\sigma^{\epsilon}(1+\delta_0)^{1-\epsilon}$ and is less than one if δ_0 is small enough.

COROLLARY 6.8: *There exists a positive Lebesgue measure subset $S \subset W^u(q)$, $N \in \mathbb{N}$ and $\lambda < 1$ such that $\forall x \in S$:*

$$\forall n > N \quad \prod_{i=0}^{n-1} \|Df_{|E^{cs}(f^i(x))}\| \leq \lambda^n.$$

The points of S are not necessarily regular in the sense of Lyapunov. We cannot use Pesin theory directly for the existence of invariant manifolds and absolute continuity of their holonomy. By dominated splitting and non-uniform hyperbolicity as above we can construct local stable manifolds.

PROPOSITION 6.9: *Every point of S has a stable manifold whose size is bounded away from zero.*

Proof: We can construct local invariant disks using only the domination property, but in the general case we do not know whether these disks are stable manifolds or not. For $f \in \mathcal{V}$ by Corollary 6.8, we are able to prove that the disks passing through the point of S are stable manifolds.

Denote $Emb(D^u, M)$ the space of C^1 embeddings from D^u to M endowed with the C^1 topology, where D^u is the u-dimensional ball of radius one.

Using notation of [7], M is an immediate relative pseudo-hyperbolic set for f if there exists a continuous function ρ such that

(3) $$\|Df_{|E^{cs}(x)}\| < \rho(x) < m(Df_{|E^{cu}(x)}),$$

where $m(T) = \|T^{-1}\|^{-1}$

In our case, dominated splitting and compactness of $\mathbb{T}^n$ imply relative pseudo-hyperbolicity. We deduce that there exist continuous sections

- ϕ^u: $M \to Emb(D^u, M)$,
- ϕ^s: $M \to Emb(D^s, M)$,

such that $W^{cs}_\epsilon(x) := \phi^s(x)D^s_\epsilon$ and $W^{cu}_\epsilon(x) := \phi^s(u)D^u_\epsilon$ have the following properties:

- $T_x W^{cs}_1(x) = E^{cs}(x)$,
- $T_x W^{cu}_1(x) = E^{cu}(x)$.

The other important property is local invariance. That is, for all $0 < \epsilon_1 < 1$ there is $0 < \epsilon_2 < 1$ such that

- $f(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f(x))$,
- $f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-1}(x))$.

Given any c we can take ϵ_1 such that

(4) $$1 - c < \frac{\|Df_{|T_y W^{cs}(x)}\|}{\|Df_{|E^{cs}(x)}\|} < 1 + c \quad \text{when } d(x,y) < \epsilon_1,\ y \in W^{cs}_{\epsilon_1}(x).$$

We can take this ϵ_1 uniformly in x as M is compact and the section is continuous with image in embeddings endowed with C^1 topology. Choosing ϵ_2 such that $f^i(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f^i(x))$ for all $0 \le i \le N$ we show that for all natural n, $d(f^n(x), f^n(y)) \le \epsilon_1$. In fact, we prove by induction that $d(f^n(x), f^n(y))$ goes to zero as n goes to infinity. Define

(5) $$\overline{\lambda} := (1 + c)\lambda$$

and c is adjusted such that $\overline{\lambda} < 1$. As $d(f^i(x), f^i(y)) \leq \epsilon_1$ for $0 \leq i \leq N+k-1$ we have

$$\begin{aligned} d(f^{N+k}(x), f^{N+k}(y)) &\leq (1+c)\|Df_{|E^{cs}(f^{N+k-1}(x))}\| d(f^{N+k-1}(x), f^{N+k-1}(y)) \\ &\leq \prod_{i=0}^{N+k-2} \|Df_{|T_{z_i}W^s_{\epsilon_1}(f^i(x))}\| \|Df_{|E^{cs}(f^{N+k-1}(x))}\| d(x,y) \\ &\leq (1+c)^{n+k} \prod_{i=0}^{N+k-1} \|Df_{|E^{cs}(f^i(x))}\| d(x,y) \leq \overline{\lambda}^{n+k} d(x,y), \end{aligned}$$

where $z_i \in W^{cs}_{\epsilon_1}(f^i(x))$; this is all by the Mean Value Theorem. ∎

7. Absolute continuity

In this section we prove that the holonomy map by the local stable manifolds constructed for the points in S is absolutely continuous.

THEOREM 4: *For large n, the holonomy map from $S \subset W^u_{local}(q(f), f)$ to $f^n(\mathcal{D}^i_\infty)$ is absolutely continuous. That is, it sends the non-zero measure subset of S to a non-zero measure subset of $f^n(\mathcal{D}^i_\infty)$.*

Let us mention that holonomy map h is defined on the whole of S for large n. From now on we call its inverse π, which is a holonomy along stable manifolds from $f^n(\mathcal{D}^i_\infty)$ to $W^u(q)$. We are going to prove that if B is a measure zero set in $h(S) \subset f^n(\mathcal{D}^i_\infty)$ then $Leb(\pi(B)) = 0$, and then conclude that $Leb(h(S)) \neq 0$. For this, it is enough to show that for every disk $D \subset f^n(\mathcal{D}^i_\infty)$ with center in $h(S)$, the holonomy π from D to $W^u(q)$ does not increase measures more than a constant which is uniform for all such disks:

$$Leb(\pi(D)) < K Leb(D).$$

Indeed, for any measurable set B with zero measure, we can cover it by a family of disks $\mathcal{D}$ such that $\sum_{D\in\mathcal{D}} m(D)$ is arbitrary close to zero. As $Leb(\pi(D)) \leq K Leb(D)$, we conclude that $Leb(\pi(B)) = 0$. From now on S' represents $h(S)$.

To prove this absolute continuity result we use the ideas of [14]. The difference is that here the points for which we construct stable manifolds are not necessarily regular. We see that a non-uniform hyperbolicity and a good control on the angles of two invariant sub-bundles is enough to get an absolute continuity result. A short sketch of the proof is as follows.

To compare $Leb(D)$ and $Leb(\pi(D))$ we iterate sufficiently such that $f^n(D)$ and $f^n(\pi(D))$ "become near enough". But after such iteration, $f^n(D)$ may have a strange shape, so in 7.2 we consider a covering of $f^n(S') \cap f^n(D)$ by $B_i := B(a_n, f^n(x_i))$ (ball of radius a_n with center $f^n(x_i)$) where x_i is in S' and a_n is much larger than $d(x_i, \pi_n(x_i))$, where π_n is defined naturally by $\pi_n = f^n \circ \pi \circ f^{-n}$.

By the specific choice of a_n, in 7.3 we show that $Leb(B_i) \approx Leb(\pi_n(B_i))$. Indeed, the dominated splitting of the tangent bundle allows us to choose them in such a good way. Finally, in 7.4 we prove some distortion results and come back to compare the volume of D and $\pi(D)$.

7.1. Some General Statements. Let us fix some notations and definitions:

- d_1 (resp. d_2) := restriction of the Riemannian metric of manifold to $f^n(D)$ (resp. $W^u(q)$).
- d_s := intrinsic metric of stable manifolds.
- d := Riemannian metric of the manifold M.
- $a \preceq b$ means $a \leq kb$ for a uniform $k > 0$, $a, b \in \mathbb{R}$.
- $a \approx b$ means that $k^{-1} < a/b < k$ for a uniform $k > 0$.

Definition 7.1: If E, F are two subspaces of the same dimension in $\mathbb{R}^n$, we define the angle between them $\measuredangle(E, F)$ as the norm of the following linear operator:

$$L\colon E \to E^{\perp} \quad \text{such that } Graph(L) := \{(v, L(v)), v \in E\} = F.$$

Definition 7.2: A thin cone $C_\epsilon(E)$ with angle ϵ around E is defined as subspace S s.t. $\measuredangle(S, E) \leq \epsilon$.

By the definition of cones it is easy to see that:

Lemma 7.3: *If $C^{cu}_\epsilon, C^{cs}_\epsilon$ are two cone fields which contain E^{cu}, E^{cs} (the subbundles of dominated splitting), then $Df_x C^{cu}_\epsilon(x) \subset C^{cu}_{\lambda\epsilon}(f(x))$ for some $0 < \lambda < 1$, or in other words the angle will decrease exponentially.*

Proof: Take $S \in C^{cu}(x)$ and $v \in S$. By definition $v = v_1 \oplus v_2$ where $v_1 \in E^{cu}, v_2 \in E^{cs}$ and by dominated splitting (see section 2)

$$\frac{||Df_x(v_2)||}{||Df_x(v_1)||} \leq \lambda \frac{||v_2||}{||v_1||},$$

and this means that $\measuredangle(Df(S), E^{cu}(f(x))) \leq \lambda \measuredangle(S, E^{cu}(x))$ by Definition 7.1. ∎

Let us state a lemma that gives us some good relations between d_1, d_2 and d.

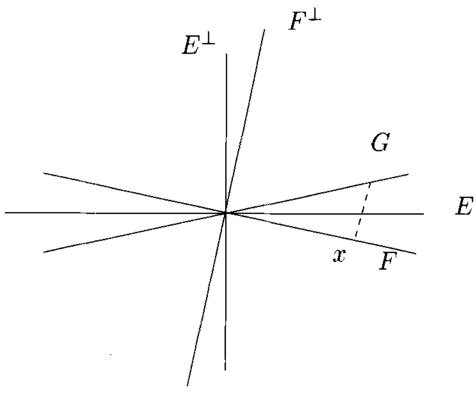
LEMMA 7.4: *Let $\mathbb{R}^n = S \oplus U$ ($U = S^\perp$) and h be a C^1 function from $B(0,\delta) \subset S$ (ball of radius δ) to F, where F is in a small cone $C_\epsilon(U)$. Suppose that $T_x(graph(h)) \subset C_\epsilon(S), \forall x \in graph(h)$. Then:*

- *$d_h(z,0) \leq C(\epsilon)d(z,0)$, where d_h is distance on $graph(h)$,*
- *$Leb(graph(h)) \leq C(\epsilon)Leb(B(0,\delta))$,*

where $C(\epsilon) \to 1$ when ϵ goes to zero.

Proof: By the hypothesis on $graph(h)$ and the definition of angle, we conclude that $||D_x h|| \leq \epsilon$ and the proof of the first item goes just by the Mean Value Theorem. The second item is also easy to prove just by the formula of the volume for the graph of a function (see Chapter 1 of [6] for the formulas). ∎

In what follows we consider a C^1 function which is defined on a ball of a linear subspace of $\mathbb{R}^n$ to another subspace. We show the relation between the norm of the derivative of such a function and another one which locally has the same graph and is defined on a slightly perturbed domain or codomain.



LEMMA 7.5: *If h is a C^1 function from $B(0,r) \subset E$ to F such that $||Dh(x)|| \leq a$ (small), where F is a linear subspace with $\measuredangle(E^\perp, F) \leq b$ (also small); then $graph(h)$ will be the graphic of a new function $\tilde{h}$: $Dom(\tilde{h}) \subset E \to E^\perp$ and $||D\tilde{h}|| \leq Ka$ (where K is a constant converging to 1 when b goes to zero).*

LEMMA 7.6: *If h is a C^1 function from $B(0,r) \subset E$ to $E^\perp$ such that $||Dh(x)|| \leq a$ (small positive number) and F is a linear subspace with the same dimension of E, with $\measuredangle(E,F) \leq b$ (also small), then $graph(h)$ will be the graphic of a new function $\tilde{h}$: $Dom(\tilde{h}) \subset F \to F^\perp$ and $||D\tilde{h}|| \leq 2(a+b)$.*

The proof of Lemma 7.5 comes out just by the definition of angle and the derivative of a function. We prove Lemma 7.6 as follows.

Proof: First observe that by the definition of angle in Definition 7.1, $\|Dh(x)\|$ is equal to the angel between E and $T_{(x,h(x))}graph(h)$. So to prove the lemma suppose that $\measuredangle(E,F) = b$ and $\measuredangle(E,G) = a$ with a, b small. Let f be a linear map from F to $F^\perp$ whose graph is E and $\tilde{g}$: $E \to F^\perp$ and g: $F \to F^\perp$ be the maps with G as their graph. We are going to show that $\|g\| \leq 2(a+b)$. By the definition of $\measuredangle(F,E)$ and using Lemma 7.5 we have (see Definition 7.1)

$$\|g(x)\| \leq \|f(x)\| + \|\tilde{g}(x + f(x))\| \leq \|f(x)\| + Ka\|x + f(x)\|,$$

where K is near to one and is obtained by Lemma 7.5. Hence we get

$$\frac{\|g(x)\|}{\|x\|} \leq b + Ka(\sqrt{1+b^2}) \leq 2(a+b)$$

and the proof of Lemma 7.6 is complete just by taking $G = T_{(x,h(x))}graph(h)$. ∎

The dependence of invariant sub-bundles E^{cu}, E^{cs} on the base point is an important support for the proof of Theorem 4. The following control of the angles is a product of dominated decomposition and can be done with the same arguments as in [15], pages 45–46.

LEMMA 7.7: *There exist constants $0 < \alpha < 1$ and $0 < \theta < 1$ with the following property:*

if $d(f^i(x), f^i(y)$ is small for $i = 0, \dots, n$ then for any two subspaces S_1, S_2 respectively in $C^{cu}(x), C^{cu}(y)$ (small cones)

$$\measuredangle(Df_x^n(S_1), Df_y^n(S_2)) \preceq \theta^n + dist(f^n(x), f^n(y))^\alpha.$$

Remark 7.8: In Lemma 7.7, $\theta < 1$ comes from dominated splitting and we can take $\theta^2 = \lambda$ where λ is as in Lemma 7.3.

7.2. COVERING $f^n(S')$ BY A GRAPH OF C^1 FUNCTIONS. We are going to show that for every point x in $S' \cap D$, $f^n(D)$ locally can be seen as the graph of a C^1 function from $E^{cu}(f^n(x))$ to $E^{cs}(f^n(x))$ with norm of derivative converging to zero uniformly as n goes to infinity. By this, we intend to cover $f^n(S') \cap f^n(D)$ by flat disks. Let us set $y_n := f^n(x)$ and $y'_n := \pi_n(y_n)$.

We mention that for all n, $f^n(D)$ is tangent to a thin cone which varies continuously. We show that there is a disk (inside $f^n(D)$) around y_n which can be described as the graph of a C^1 function. The size of this disc decays when n grows, but it is definitely larger than the stable distance (d_s) between y_n and y'_n.

LEMMA 7.9: *There exists $\delta > 0$ such that for $\delta_1 < \delta$ and any $x \in M$, if*

$$h\colon B^{cu}_{\delta_1}(0) \subset E^{cu}(x) \to E^{cs}(x), \quad h(0) = 0, \quad \|Dh(\xi)\| \le k, \quad \forall \xi \in B^{cu}_{\delta_1}$$

and graph $(h) \subset B^{cu}_{\delta} \times B^{s}_{\delta}$, then $W = f(graph(h)) \cap (B^{cu}_{\gamma\delta_1} \times B^{s}_{\delta})$ will also be a graph of some $\tilde{h}$ with the following properties:

1. *its domain contains $B^{cu}_{\gamma\delta_1}$ and $\tilde{h}(0) = 0$;*
2. *$\|D\tilde{h}(\tilde{\xi})\| \le k\theta, \forall \tilde{\xi} \in B^{cu}_{\gamma\delta_1} \subset E^{cu}(f(x))$;*
3. *$\bar{\lambda} < \gamma$ where $\bar{\lambda}$ is defined in equation (5) in Section 6.*

Proof: As f is C^2, there exists δ such that for all $x \in M$, f can be written as

$$f(\xi, \eta) = (A^{cu}(\xi) + \phi^{cu}(\xi, \eta), A^{cs}(\eta) + \phi^{cs}(\xi, \eta),$$

where $(\xi, \eta) \in B^{cu}_{\delta} \times B^{cs}_{\delta}$ and $\|D(\phi^{cu}, \phi^{cs})\| \le \epsilon$. Just to reduce the notations suppose that x is a fixed point. We define

- $\alpha(\xi) = \tilde{\xi} := A^{cu}(\xi) + \phi^{cu}(\xi, h(\xi) = A^{cu}(\xi + (A^{cu})^{-1}(\xi)\phi^{cu}(\xi, h(\xi));$
- $\beta(\xi) = A^{cs}(h(\xi)) + \phi^{cs}(\xi, h(\xi))$; $\xi \in B^{cu}_{\delta}(0)$.

Now as $\|(A^{cu})^{-1}\| \le 1 + \delta_0$, choosing ϵ small enough we deduce that

$$\|(A^{cu})^{-1}\| Lip(\phi^{cu}(\xi, h(\xi))) < 1,$$

and this shows that $\alpha = A^{cu}(.)(Id + (A^{cu})^{-1}(.)\phi^{cu}(., h(.))$ is invertible. So it is enough to determine the domain of α^{-1} and define $\tilde{h} = \beta \circ \alpha^{-1}$ for proving the first part of the lemma.

Observe that

$$\|\alpha(\xi)\| \ge \|A^{cu}(\xi)\| - \|\phi^{cu}(\xi, h(\xi)\| \ge \Big(\frac{1}{1+\delta_0} - 2\epsilon\Big)\|\xi\| > \gamma\|\xi\|,$$

where γ is near to one as δ_0 is small enough. Now with the aid of the proof of the inverse function theorem, α^{-1} is defined on $B^{cu}_{\gamma\delta_1}$ and

$$\tilde{h} = \beta \circ \alpha^{-1}\colon B^{cu}_{\gamma\delta_1} \to B^{cs}$$

is what we want. Observe that as $\bar{\lambda} < 1$, the third part of the lemma also turns out.

Now we will verify the claim about the derivative of $\tilde{h}$. By dominated splitting we have $0 < \theta < 1$ such that $\|(A^{cu})^{-1}(f(x))\|\|A^{cs}(x)\| \le \theta^2$ (θ^2 is just the λ in Lemma 7.3). By choosing ϵ small enough such that $\|D\beta\| \le k/\sqrt{\theta}$ we get

$$\|D\tilde{h}(\tilde{\xi})\| \le \|D\beta(\xi)\|\|D\alpha^{-1}(\tilde{\xi})\| \le \frac{k}{\sqrt{\theta}}\|A^{cs}\|\|(A^{cu})^{-1}\|\|D(I+T)^{-1}\|,$$

where $T = (A^{cu})^{-1}\phi^{cu}(\xi, h(\xi))$. On the other hand, we have

$$\|D(I+T)^{-1}\| = \|(I+DT)^{-1}\| \le \sum_{i=0}^{\infty}\|(DT)^i\| = \frac{1}{1-\|DT\|} \le \frac{1}{\sqrt{\theta}}$$

for ϵ small enough. So $\|D\tilde{h}(x)\| \le k\theta$. ∎

Let us see how to cover $f^n(S') \cap f^n(D)$ by disks. For $x \in S' \cap D$ there exists $\delta > 0$ (uniform in D) and C^1 functions h_x such that $h_x\colon E^{cu}_{\delta}(x) \to E^{cs}(x)$ and the graph of h_x is a ball around x. Now by Lemma 7.9 there exists $h_{f^n(x)}\colon B^{cu}_{\gamma^n\delta}(x) \to E^{cs}(f^n(x))$ such that $h_{f^n(x)}(B^{cu}_{\gamma^n\delta}(x))$ is a ball around y_n and also we have a good control on their derivative. That is,

$$\|Dh_n(x)\| \le k\theta^n$$

where h_n represents any $h_{f^n(x)}$. Applying Lemma 7.4 we get

$$d(z, y_n) \le d_1(z, y_n) \le k_n d(z, y_n) \quad \forall z \in \operatorname{graph}(h_n) \text{ and } k_n \to 1.$$

So we conclude that $h_{f^n(x)}(B^{cu}_{\gamma\delta}(x))$ is a ball of radius arbitrarily near to $2a_n := \gamma^n\delta$ by taking n large enough. We call this ball $\bar{B}_n$ (around y_n) and B_n the ball with radius a_n around y_n.

We mention that $\bar{B}_n$ is also the graph of a function from E^{cu} to $(E^{cu})^{\perp}$ over $P(\bar{B}_n)$, where P is the orthogonal projection along $(E^{cu})^{\perp}$.

Remark 7.10: By the estimate of the derivative of h_n, $P(\bar{B}_n)$ is contained in the ball of radius $2a_n(1 + C\theta^n)$ and contains the ball of radius $2a_n(1 - C\theta^n)$, where C depends on the angle of $(E^{cu})^{\perp}$ and E^{cs}.

In what follows we are working with $\bar{B}_n$ as the graph of the mentioned new C^1 function, which we also call h_n, and it is easy to see that $\|Dh_n\| \le K\theta^n$ (Lemma 7.5).

Now we define a new transformation from $\bar{B}_n$ to $W^u(q)$ which is very near to holonomy π_n. Let $z \in \bar{B}_n$ and define $\mathcal{P}(z)$ by translation along $E^{cu}(y_n)^{\perp}$ which is orthogonal to the tangent space of all points of $\bar{B}_n$. One important property of $\mathcal{P}$ is that $d(z, \mathcal{P}(z))$ is exponentially small. Indeed, we choose a_n small enough for $d(z, \mathcal{P}(z))$ to be comparable to $d(y_n, y_n') = \bar{\lambda}^n$.

7.3. Comparing measures of B_i and $\pi_n(B_i)$. In the previous section we saw how to cover $f^n(S') \cap f^n(D)$ by balls $\bar{B}_i$. In what follows we prove that the volume of these disks does not increase "a lot" by holonomy. Indeed, we have to take a_n in a good way to have this property. The most important property for a_n is

$$\bar{\lambda}^n / a_n \to 0 \tag{6}$$

and the main proposition is the following.

Proposition 7.11: *There is a constant $I > 0$, independent of n, such that $Leb(\pi_n(B_i)) \leq I Leb(B_i)$*

To prove the above Proposition, we start with some lemmas.

Lemma 7.12: *There is a choice of a_n satisfying (6) such that for every $z \in \bar{B}_i$, $d(z, \mathcal{P}(z)) \preceq \bar{\lambda}^n$.*

Proof: When n is large enough we can consider $\mathcal{P}(\bar{B}_n)$ also as a graph over $E^{cu}(y_n)$ to $E^{cu}(y_n)^{\perp}$, but we have to consider the angle between $E^{cu}(y_n)$ and $E^{cu}(y_n')$ to calculate the norm of the derivative of the new function. To estimate the norm of the derivative of the C^1 functions whose graphs are $\bar{B}_n$ and $\mathcal{P}(\bar{B}_n)$, we use Lemmas 7.6 and 7.5. Using the Mean Value Theorem and Remark 7.10, we have (see Figure 3)

$$\begin{aligned} d(z, \mathcal{P}(z)) \leq & K(2a_n + 2Ca_n\theta^n)\theta^n \\ & + (2a_n + 2Ca_n\theta^n)(K\theta^n + \measuredangle(E^{cu}(y_n), E^{cu}(y_n'))) + \bar{\lambda}^n. \end{aligned}$$

Note that the term containing angles in the above relations arises because of the deviation of $E^{cu}(y_n)$ from $E^{cu}(y_n')$ and, applying Lemma 7.6,

$$\measuredangle(E^{cu}(y_n), E^{cu}(y_n')) \preceq \theta^n + d(y_n, y_n')^{\alpha},$$

and therefore

$$d(z, \mathcal{P}(z)) \preceq a_n\theta^n + a_n d(y_n, y_n')^{\alpha} + \bar{\lambda}^n.$$

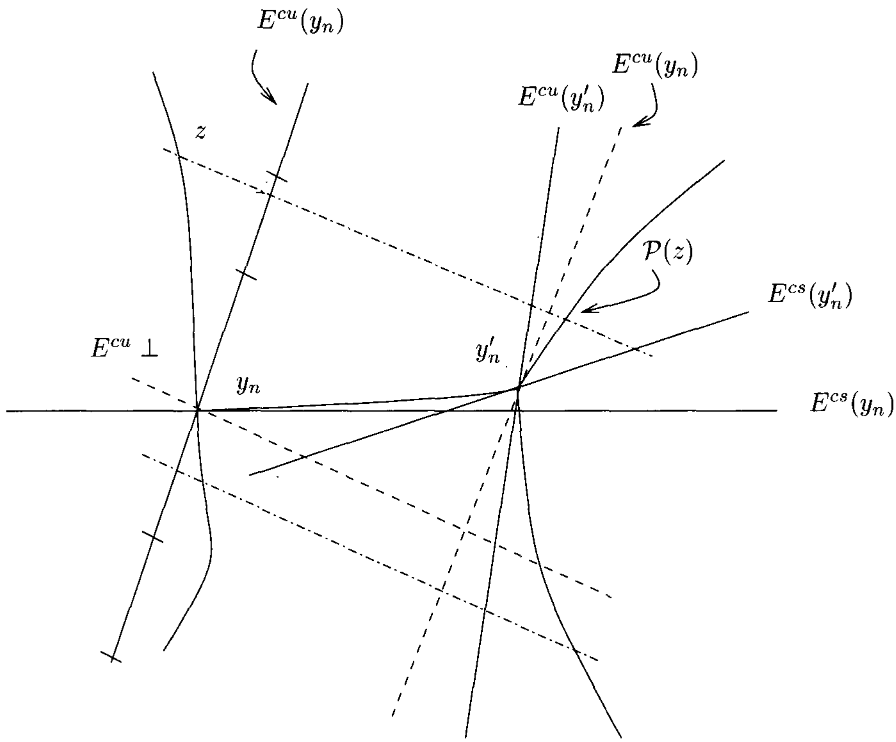


Figure 3. Graphs over E^{cu}.

So to finish the proof of Lemma 7.12 it is enough to choose a_n satisfying the following two conditions:

- $a_n \approx \bar{\lambda}^n \theta^{-n}$,
- $a_n \approx \bar{\lambda}^{n(1-\alpha)}$.

Remember that by Lemma 7.9, we need another restriction on a_n to have a graph of functions to use Mean Value Theorem:

- $a_n \leq \gamma^n \delta$.

So choose $a_n = \min(\bar{\lambda}^n \theta^{-n}, \bar{\lambda}^{n(1-\alpha)}, \gamma^n \delta)$. As $\bar{\lambda} < \gamma$, already $\bar{\lambda}^n / a_n \to 0$. ∎

LEMMA 7.13: *$\pi_n(B_i)$ is contained in a ball around y'_n of radius near enough to $\frac{3}{2}a_n$ as n is large enough.*

Proof: For $z \in B_i$, $\pi_n(z)$ lies in $W^u(q)$, which is contained in the graph of a function defined globally, and the graph is tangent to a thin cone field. So by

Lemma 7.4 we deduce that for $z \in B_i$, $d_2(\pi_n(z), y'_n) \le \frac{3}{2} d(\pi_n(z), y'_n)$ and

$$\begin{aligned} d_2(\pi_n(z), y'_n) &\le \frac{3}{2}(d(\pi_n(z), y'_n)) \le \frac{3}{2}(d(\pi_n(z), z) + d(z, y_n) + d(y_n, y'_n)) \\ &\le \frac{3}{2}(d_s(\pi_n(z), z) + d_1(z, y_n) + d_s(y_n, y'_n) \le \frac{3}{2}(\bar\lambda^n + k_n a_n + \bar\lambda^n) \\ &= \frac{3}{2} a_n \Big(k_n + \frac{2\bar\lambda^n}{a_n}\Big). \end{aligned}$$

So by choosing a_n as in Lemma 7.12 the proof is complete. ∎

Lemma 7.14: *$\mathcal{P}(\bar B_i)$ contains $\pi_n(B_i)$.*

Proof: For every $z \in \bar B_i$, by triangular inequality we have

$$\begin{aligned} d_2(\mathcal{P}(z), y'_n) &\ge d(\mathcal{P}(z), y'_n) \ge d(z, y_n) - d(y'_n, y_n) - d(z, \mathcal{P}(z)) \\ &\ge \frac{1}{k_n} d_1(z, y_n) - d_s(y'_n, y_n) - d(z, \mathcal{P}(z)) \ge \frac{1}{k_n} d_1(z, y_n) - 2\bar\lambda^n. \end{aligned}$$

Here we have used Lemma 7.12, as $k_n \to 1$ and $\bar\lambda^n / a_n \to 0$. We conclude that $\mathcal{P}(\bar B_i)$ contains a ball with radius near to $2a_n$ and by Lemma 7.13 it contains $\pi_n(B_i)$. ∎

Proof of Proposition 7.11: Choose a_n as in Lemma 7.12. As $Leb(\bar B_i) \le I_1 Leb(B_i)$ for a constant I_1 not depending on n and just depending on the dimension of B_i, we have

$$Leb(\pi_n(B_i)) \le Leb(\mathcal{P}(\bar B_i)) \approx Leb(\bar B_i) \le I_1 Leb(B_i)$$

and the proposition is proved. ∎

Up to now we have covered $S_n := f^n(S') \cap f^n(D)$ by a family of disks such that the volume of whose images under holonomy is comparable to their volume. By the Besicovich covering theorem [10] we can cover S_n with a countable locally finite subfamily $\{B_i\}_i$; that is, there is a constant C only depending on the dimension of D such that the intersection of any $C+1$ disk of such a subfamily is the empty set.

7.4. Distortion Estimates. Now we state the distortion control statements. By $Jf(x, A)$ we mean $det(Df_x|A)$.

LEMMA 7.15 (Bounded Distortion): *There are $P_1, M > 0$ such that for any $z \in B_i$ the following are satisfied:*

- $\frac{1}{M} \leq \frac{Jf^{-n}(y_n, T_{y_n} B_i)}{Jf^{-n}(y'_n, T_{y'_n} \mathcal{P}(B_i))} \leq M,$
- $\frac{1}{P_1} \leq \frac{Jf^{-n}(z, T_z(B_i))}{Jf^{-n}(y_n, E^{cu}_{y_n})} \leq P_1.$

Proof: The problem is that in general we do not have Hölder control of the center unstable fibers. But in the case of dominated decomposition or, in other words, when we have a hyperbolicity property for the angles, one can show statements near to Hölder continuity.

As f is a C^2 function, we conclude that there exist constants $R_1, R_2 > 0$ such that if $z_1, z_2 \in M, d(z_1, z_2) \leq 1$ and S_1, S_2 are subspaces of $\mathbb{R}^n$ with dimension u (dimension of E^{cu}), then

$$(7) \qquad |\log Jf^{-1}(z_1, A_1) - \log Jf^{-1}(z_2, A_2)| \leq R1 d(z_1, z_2) + R_2 \measuredangle(A_1, A_2).$$

Now using the above inequality and Lemma 7.7 we have

$$\begin{aligned}
&|\log Jf^{-n}(y_n, E^{cu}(y_n)) - \log Jf^{-n}(y'_n, E^{cu}(y'_n))| \\
\leq & R_1 \left(\sum_{i=0}^{n-1} dist(f^{-i}(y_n), f^{-i}(y'_n)) \right) + R_2 \left(\sum_{i=0}^{n-1} \measuredangle(E^{cu}(f^{-i}(y_n)), E^{cu}(f^{-i}(y'_n))) \right) \\
\leq & \frac{CR_2}{1-\theta} + (KR_2 + R_1) \sum_{i=0}^{n-1} dist(f^{-i}(y_n), f^{-i}(y'_n))^\alpha
\end{aligned}$$

for some constants $C, K > 0$. So again using (7) we conclude

$$\begin{aligned}
&|\log Jf^{-n}(y_n, T_{y_n} B_i) - \log Jf^{-n}(y'_n, T_{y'_n} \mathcal{P}(B_i))| \\
\leq & |\log Jf^{-n}(y_n, T_{y_n} B_i) - \log Jf^{-n}(y_n, (E^{cu}(y_n)))| \\
& + |\log Jf^{-n}(y_n, E^{cu}(y_n)) - \log Jf^{-n}(y'_n, E^{cu}(y'_n))| \\
& + |\log Jf^{-n}(y'_n, E^{cu}(y'_n)) - \log Jf^{-n}(y'_n, T_{y'_n} \mathcal{P}(B_i))| \\
(8) \qquad \leq & \frac{R_2}{1-\theta} + (KR_2 + R_1) \sum_{i=0}^{n-1} dist(f^{-i}(y_n), f^{-i}(y'_n))^\alpha + 2R_2 \sum_{i=0}^{n-1} \theta^{n-i}.
\end{aligned}$$

As y_n, y'_n are on the same strong stable manifold, all of the terms appearing in (8) are summable and the proof of the first item of the lemma is complete. In fact, our argument shows that we can substitute y_n, y'_n respectively by any point $w_n \in B_i \cap f^n(S')$ and $\pi_n(w_n)$. The second item of the lemma comes from the same arguments, remembering that the size of $B_i \subset f^n(D)$ is exponentially small. ∎

Now we apply distortion estimates of Jacobians to get

$$Leb(\pi(D)) \leq \sum_i Leb(f^{-n}(\pi_n(B_i)) \leq MP_1^2 \sum_i Leb(f^{-n}(B_i)) \frac{Leb(\pi_n(B_i))}{Leb(B_i)}$$
$$\leq IMP \sum_i Leb(f^{-n}(B_i)).$$

But as $\{B_i\}_i$ is a locally finite family covering S_n and by f^{-n}, the areas of disks tangent to C^{cu} decreases. Taking n sufficiently large we see that $\sum_i Leb(f^{-n}(B_i)) \leq ALeb(D)$. So taking $P_1^2 = P$ we conclude

$$\frac{Leb(\pi(D))}{Leb(D)} \leq IMPA(universal).$$

8. Appendix A: Robust indecomposability

Topological transitivity of C^1 diffeomorphisms and ergodicity (metric transitivity) of the Lebesgue measure for the C^2 conservative systems are two kinds of indecomposability. The existence of SRB measures with full support and full Lebesgue measure of the basin (like in the C^2-Anosov diffeomorphism case) is also a kind of indecomposability which in the conservative diffeomorphism case implies ergodicity. By results of [3] we know that C^1-robust transitivity implies dominated splitting. For constructing SRB measures we need more regularity than C^1. So we define C^1-robust indecomposability as follows:

Definition 8.1: Let $\mathrm{Diff}^{1+} = \bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}(M)$. For $f \in \mathrm{Diff}^{1+}$ we say f is C^1-robustly indecomposable if there is an open set $U \subset \mathrm{Diff}^1(M)$ such that any $g \in U \cap \mathrm{Diff}^{1+}$ has an SRB measure with $Leb(B(\mu)) = 1$ and $\mathrm{Supp}(\mu) = M$.

PROPOSITION 8.2: *Any C^1-robustly indecomposable diffeomorphism has dominated splitting.*

Proof: Let U be an open set as in Definition 8.1. We claim that any $f \in U \cap \mathrm{Diff}^{1+}(M)$ is transitive. To show this, take two open sets A, B in M. As $\mathrm{Supp}(\mu) = M$, so $\mu(A), \mu(B) > 0$. Let $x \in B(\mu)$; by definition of the basin, the orbit of x goes through A and B infinitely many times. This means that some iterate of A intersects B.

Now suppose $g_1 \in U$ does not admit dominated splitting; by the results in [3] one can perturb g_1 to get $g_2 \in U$ with a sink. Now by the density of $\mathrm{Diff}^{1+}(M)$ in $\mathrm{Diff}^1(M)$ and persistence of sinks in C^1 topology, we get a diffeomorphism

g_3 in $U \cap \mathrm{Diff}^{1+}(M)$ which has a sink and so cannot be transitive, contradicting the above claim. ∎

9. Appendix B: Simultaneous hyperbolic times

In [1, Theorem 6.3], ergodic *cu*-Gibbs measures for diffeomorphisms with dominated splitting and the non-uniform hyperbolic property like in the Preliminaries (Section 2) are constructed. These measures are absolutely continuous along a family of disks which are tangent to the center-unstable cone field.

PROPOSITION 9.1: *For $f \in \mathcal{V}$, the cu-Gibbs measures as above are SRB; that is, their basin has positive Lebesgue measure.*

To prove that these measures are SRB, one needs to show that for points in the support of these measures, all the Lyapunov exponents (in the E^{cs} direction) are negative. To provide negative Lyapunov exponents, in [1], the authors add the condition of "simultaneous hyperbolic times". We show that for $f \in \mathcal{V}$ it is not necessary to verify this condition and see that the *cu*-Gibbs measures constructed there are indeed SRB measure.

For any $y \in \mathrm{Supp}(\mu)$, where μ is one of such *cu*-Gibbs measures, there exists x such that $y \in D^\infty(x)$, where $D^\infty(x)$ is tangent E^{cu} at any point of it and, moreover, it is the local strong unstable manifold of x (see [1, Lemma 3.7]).

LEMMA 9.2: *If $f \in \mathcal{V}$, then for Lebesgue almost all points of $D^\infty(x)$ the Lyapunov exponents in the E^{cs} direction are negative.*

Proof: By the above observations about $D^\infty(x)$ we may consider the lift of $D^\infty(x)$ to $\mathbb{R}^n$ included in the graph of a global C^1 function $\gamma\colon \mathbb{R}^u \to \mathbb{R}^s$ with

$$T_{(z,\gamma(z))} graph(\gamma) \in C^{cu}(z, \gamma(z)).$$

So by the definition of dynamically flat submanifolds in Section 6, $D^\infty(x)$ is contained in a dynamically flat submanifold and, by Proposition 6.3, for almost all points in $D^\infty(x)$ all the Lyapunov exponents in the E^{cs} direction are negative. ∎

Now using Lemma 9.2, a standard argument shows that the *cu*-Gibbs measures are really SRB, or their basins have positive volume.

Proof of Proposition 9.1: Let μ be such a Gibbs ergodic measure. There exists some disk D^∞ such that almost every point in D^∞ is in the basin of μ. By absolute continuity of stable lamination of the points in $D^\infty \cap B(\mu)$ and the fact that these stable manifolds are contained in $B(\mu)$, we conclude that the basin of μ must have positive Lebesgue measure. ∎

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